

An Alternative Approach for the Derivation of the Fractional Black-Scholes Equation for the Pricing of Options and its Solution via the Mellin Transform

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Abstract – We derive in this contribution a fractional Black-Scholes -Merton-like valuation formulas for European power vanilla options driven by a fractional Brownian motion with Hurst index $H \in (0.5, 1)$. Furthermore, we remove the effect of the interest rate r , using a simple transformation. Then a new technique which does not require transformation of variable for the solution of the fractional Black-Scholes partial differential equation for option prices is derived using the Mellin transforms.

Keywords – Fractional Black-Scholes Partial Differential Equation, Mellin Transform, Option Price.

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I. INTRODUCTION

Fractional calculus has become of increasing use for analyzing not only stochastic processes driven by fractional Brownian processes (Wyss [1]), but also non-random fractional phenomena in physics (Jumarie[2]), like the study of porous systems, for instance, and quantum mechanics (Shawagfeh[3]). whichever the framework is, we believe that the very reason for introducing and using fractional derivative is to deal with non-differentiable functions.

In engineering applications of probability, stochastic processes are often used to model the input of a system. For instance, the financial mathematics requires stochastic models for the time evolution of assets and the queuing networks analysis is based on models of the offered traffic. Hitherto, the stochastic processes used in these fields are often supposed to be Markovian. However, recent studies (Leland et al[4]) show that real inputs exhibit long-range dependence: the behavior of a real process after a given time t does not only depend on the situation at t but also of the whole history of the process up to time t . Moreover, it turns out that this property is far from being negligible because of the effects it induces on the expected behavior of the global system (Norros[5]).

In practice, random fluctuations of interest rate over time have a significant contribution to the change of an option price. An *option* on a stock is an asymmetric contract which entitles the holder to buy (*call*) or sell (*put*) a share at a specified price (strike or exercise price) on (European) or before (American) a certain date. The basic background for option pricing is given in Sharpe et al, [6] and a more specialized treatment may be found in Wilmott et al [7]. For the last three decades, much of the mathematical study in this area has focused on the

boundary value problems associated with the Black-Scholes partial differential equation (see Black and Scholes [8], Merton [9]).

Based on this observation, some work has been reported on the price formula of European options with stochastic interest rate. Most of all, Merton [10], Rabinovitch [11], and Amin and Jarrow [12] have proposed the formula of closed form European option pricing under the Gaussian interest rate by using relatively simple algebra. This method is also discussed in detail by Kim [13]. Also, Fang [14] derived an exact pricing formula for European option under stochastic interest rate by applying martingale method. However, the closed formula for the prices of options has been studied usually by utilizing probabilistic techniques as the papers stated above. The use analytic methods based on Mellin transforms as a better way to compute the option prices had been done (Yoon [15]).

The Mellin transform is defined as an integral transform that may be considered as the multiplicative version of the two-sided Laplace transform. Many papers have shown that the Mellin transform technique would help us resolve the complexity of the calculation compared to the probabilistic approach. Panini and Srivastav [16] studied the pricing formula of a European vanilla option and a basket option using Mellin transforms. Panini and Srivastav [17] found also the pricing of perpetual American options with Mellin transforms. Frontczak and Schöbel [18] used Mellin transforms to value American call options on dividend-paying stocks. Also, Elshegmani and Ahmed [19] derived analytical solution for an arithmetic Asian option using Mellin transforms.

Fractional differential equation (FDE) can be extensively applied to various disciplines such as physics, mechanics, chemistry and engineering, (see Mainardi [20], Buckwar and Luchko[21]). Hence, in recent years, fractional differential equations have been of great interest and there have been many results on existence and uniqueness of the solutions of FDE, (see Zhu et al[22], Wang et al[23]), thus giving good motivation for further development of this topic. A fractional Black-Scholes formula for the price of an option for every $t \in [0, T]$ driven by a fractional Brownian motion is a family member of the FDE.

Our main objective this paper is to derive fractional Black-Scholes equation driven by a fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$. Furthermore, we remove the effect of the interest rate r , using a simple transformation. Then a new technique which does not require transformation of variable for the solution of the

fractional Black-Scholes partial differential equation for option prices is derived using the Mellin transforms.

II. BACKGROUND

In this section we summarize the results from Du et al [24] that we will need. Fix a Hurst constant $H, \frac{1}{2} < H < 1$.

Define

$$\phi(s, t) = H(2H - 1)|s - t|^{2H-2}; s, t \in \mathbb{R}. \quad (2.1)$$

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be measurable. Then we say that $f \in L^2_\phi(\mathbb{R})$ if

$$|f|_\phi^2 := \int_{\mathbb{R}} \int_{\mathbb{R}} f(s)f(t)\phi(s, t)dsdt < \infty. \quad (2.2)$$

If we equip $L^2_\phi(\mathbb{R})$ with the inner product

$$(f, g)_\phi := \int_{\mathbb{R}} \int_{\mathbb{R}} f(s)g(t)\phi(s, t)dsdt; f, g \in L^2_\phi(\mathbb{R}). \quad (2.3)$$

then $L^2_\phi(\mathbb{R})$ becomes a separable Hilbert space. In fact, we have

Lemma 2.1: Let

$$\Gamma_\phi f(u) = c_H \int_u^\infty (t - u)^{H-3/2} f(t)dt \quad (2.4)$$

where $c_H = \frac{H(2H-1)\Gamma(\frac{3}{2}-H)}{\Gamma(H-\frac{1}{2})\Gamma(2-2H)}$ and Γ denotes the gamma

function. Then Γ_ϕ is an isometry from $L^2_\phi(\mathbb{R})$ to $L^2(\mathbb{R})$.

Proof: By a limiting argument, we may assume that f and g are continuous with compact support. By definition,

$$\begin{aligned} & (\Gamma_\phi(f), \Gamma_\phi(g))_{L^2(\mathbb{R})} \\ &= c_H^2 \int_{\mathbb{R}} \left\{ \int_u^\infty (s - u)^{H-3/2} f(s)ds \int_u^\infty (t - u)^{H-3/2} g(t)dt \right\} du \\ &= c_H^2 \int_{\mathbb{R}} f(s)g(t) \left\{ \int_{-\infty}^{s \wedge t} (s - u)^{H-3/2} (t - u)^{H-3/2} du \right\} dsdt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(s)g(t)\phi(s, t)dsdt = (f, g)_\phi \end{aligned}$$

where we have used the identity

$$c_H^2 \int_{-\infty}^{s \wedge t} (s - u)^{H-3/2} (t - u)^{H-3/2} du = \phi(s, t),$$

(see for example Gripenberg and Norros[25], p. 404).

If $f \in L^2_\phi(\mathbb{R})$ (deterministic) one can define

$\int_{\mathbb{R}} f(t)dB_H(t) = \int_{\mathbb{R}} f(t)\delta B_H(t)$ in the usual way by first considering simple integrands

$$f_m(t) = \sum_i a_i^{(m)} \chi_{[t_i, t_{i-1}]}(t),$$

Setting

$$\int_{\mathbb{R}} f_m(t)dB_H(t) = \sum_i a_i^{(m)} (B_H(t_{i-1}) - B_H(t_i)) \quad (2.5)$$

and defining

$$\int_{\mathbb{R}} f_m(t)dB_H(t) = \lim_{m \rightarrow \infty} \int_{\mathbb{R}} f_m(t)dB_H(t). \quad (2.6)$$

The limit exists in $L^2(\mu_\phi)$ because of the isometry

$$\mathbb{E} \left(\int_{\mathbb{R}} f_m(t)dB_H(t) \right)^2 = |f_m|_\phi^2, \quad (2.7)$$

where μ_ϕ is the probability law of B_H (see also next section). For $f \in L^2_\phi(\mathbb{R})$ define

$$\varepsilon(f) = \exp \left(\int_{\mathbb{R}} f dB_H - \frac{1}{2} |f|_\phi^2 \right). \quad (2.8)$$

Then we have (Duet al [24], Theorem 3.1):

The linear span of $\{\varepsilon(f); f \in L^2_\phi(\mathbb{R})\}$ is dense in $L^2(\mu_\phi)$.

III. FORMULATION OF THE FRACTIONAL BLACK-SCHOLES EQUATION

Geometric Brownian motion is often used to model the price of a share. In this paper, our aim is to model the price of a share instead with the fractional Brownian motion with Hurst parameter H ; ($0 < H < 1$).

Let $X(t)$ denote the price of one unit of a given share at time t and suppose that the price dynamics of the share is given by

$$dX(t) = a(t)X(t)dt + b(t)X(t)dW_H(t) \quad (3.1)$$

where $a, b: [0, T] \rightarrow \mathbb{R}$ are some functions and $W_H(t)$ a fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$.

We interpret $a(t)$ as the interest rate and $b(t)$ as the volatility, which is the strength of fluctuations in the market. The price of a European call option for the share X can be found by the fractional Black Scholes equation.

Theorem 3.1 (Fractional Black-Scholes Equation):

Let the price $p(t, x)$ of a European call option for a share satisfying (3.1) with initial condition $X(t) = x$, is given by the partial differential equation

$$\begin{cases} \frac{\partial p}{\partial t} + a(t)x \frac{\partial p}{\partial x}(t, x) + H(b(t)x)^2 t^{2H-1} \frac{\partial^2 p}{\partial x^2}(t, x) \\ = a(t)p(t, x), \text{ for } 0 < t < T, x \in \mathbb{R} \\ p(T, x) = (x - K)^+ \text{ for } x \in \mathbb{R}. \end{cases} \quad (3.2)$$

Here K is the pre-specified price, for which the owner of the option may purchase one unit of the stock X at time T .

Proof: We prove this theorem in the case of constant a and b for simplicity (though the proof is similar for non-constant a and b). Let X be given by (3.1) and assume the existence of a risk free paper B satisfying.

$$dB(t) = aB_H(t)dt. \quad (3.3)$$

Consider now the portfolio replicating the option

$$I(t) = -p(t, X(t)) + \alpha(t)X(t) + \beta(t)B_H(t), \quad (3.4)$$

for some $\alpha(t), \beta(t) \in \mathbb{R}$. In the sense that no money is brought in or taken out of the portfolio, we say that the portfolio self-financing. That is applying the Itô formula to $I(t)$ (and suppressing the evaluation point $(t, X(t))$ for notational clarity), we obtain

$$\begin{aligned} dI(t) &= -dp + d(\alpha(t)X(t) + \beta(t)B(t)) \\ &= \frac{\partial p}{\partial t} dt - \frac{\partial p}{\partial x} dX(t) - \frac{\partial^2 p}{\partial x^2} D_t^\emptyset x(t) \beta(t) dX + \alpha(t) dX(t) \\ &\quad + \beta(t) dB(t) \end{aligned}$$

$$\begin{aligned} &= \left(-\frac{\partial p}{\partial t} - \frac{\partial p}{\partial x} aX(t) - \frac{\partial^2 p}{\partial x^2} H b^2 t^{2H-1} (X(t))^2 + \right. \\ &\quad \left. \alpha(t)X(t) + \beta(t)aB(t) \right) dt + \left(\alpha(t) - \frac{\partial p}{\partial x} \right) bX(t) dW_H(t) \end{aligned} \quad (3.5)$$

where $D_t^\phi X(t) = bX(t) \int_0^t \phi(t, u) du = bHX(t)t^{2H-1}$.

For the portfolio to become riskless choose $\alpha(t) = \frac{\partial p}{\partial x}$ so that

$$dI(t) = \left(-\frac{\partial p}{\partial t} - \frac{\partial^2 p}{\partial x^2} Hb^2 t^{2H-1} (X(t))^2 + \beta(t) aB(t) \right) dt. \quad (3.6)$$

Suppose that the opportunity of arbitrage is made impossible, so that the probability, of gaining money without risk is zero, then $I(t)$, being riskless, must evolve as the risk free paper B.

Hence $dI(t) = aI(t)dt$ and, by combining (3.4) and (3.6), we have

$$\begin{aligned} & \left(-\frac{\partial p}{\partial t} - \frac{\partial^2 p}{\partial x^2} Hb^2 t^{2H-1} (X(t))^2 + \beta(t) aB(t) \right) dt \\ &= dI(t) = aI(t)dt = a(-p + \alpha(t)X(t) + \beta(t)B(t))dt \\ &= a \left(-p + \frac{\partial p}{\partial x} X(t) + \beta(t)B(t) \right) dt. \end{aligned} \quad (3.7)$$

Setting $x = X(t)$, we have the fractional partial differential

$$\frac{\partial p}{\partial t} + ax \frac{\partial p}{\partial x} + \frac{\partial^2 p}{\partial x^2} Hb^2 t^{2H-1} x^2 - ap = 0. \quad (3.8)$$

For $t < T$, where T is the exercise time of the option. At time T the cost of exercising the option is K so the terminal condition corresponding to (3.8) must be

$$p(T, x) = (x - K)^+. \quad (3.9)$$

We now represent the solution to the following partial differential equation with prescribed terminal value

$$\left(\begin{aligned} \frac{\partial p}{\partial t}(t, x) + AP(t, x) &= 0, \text{ for } 0 < t < T, x \in \mathbb{R} \\ p(T, x) &= \phi(x), \text{ for } x \in \mathbb{R} \end{aligned} \right), \quad (3.10)$$

where the differential operator A is given by

$$AP(t, x) = \mu(t, x) \frac{\partial p}{\partial x}(t, x) + H(\sigma(t, x))^2 t^{2H-1} \frac{\partial^2 p}{\partial x^2}(t, x) \quad (3.11)$$

In terms of the fractional stochastic differential equations, let P be a solution to (3.10), fix the point (t, x) and define X to be the fractional differential equation $dX(t) = \mu(t, X(s))dt + \sigma(t)X(s)dW_H(t)$, $\mu, \sigma \in L_{\phi}^{1,2}$,

$$(3.12)$$

where σ is the volatility, μ the drift parameter with initial value $X(t) = x$. if $p \in C^2(\mathbb{R}_+ \times \mathbb{R})$ then we have;

$$\begin{aligned} (T, X(t)) &= P(t, X(t)) + \int_t^T \frac{\partial P}{\partial s}(s, X(s))ds + \\ & \int_t^T \frac{\partial p}{\partial x}(s, X(s))\mu(s)ds + \int_t^T \frac{\partial p}{\partial x}(s, X(s))\sigma(s)dW_H + \\ & \int_t^T \frac{\partial^2 p}{\partial x^2}(s, X(s))\sigma(s)D_s^\phi ds = \int_t^T \left(\frac{\partial p}{\partial s}(s, X(s)) + \right. \\ & \left. AP(s, X(s)) \right) ds = \int_t^T \frac{\partial p}{\partial x}(s, X(s))\sigma(s)dW_H, \end{aligned} \quad (3.13)$$

P is the solution to (3.10), so (3.13) becomes

$$\phi(X(t)) - P(t, x(t)) = \int_t^T \frac{\partial^2 p}{\partial x^2}(s, X(s))dW_H(s). \quad (3.14)$$

Taking expectation of both sides gives

$$p(t, x) = E^{t,x}[\phi(X(T))], \quad (3.15)$$

Where the indices t, x means that the process X satisfies $X(t)$. Similarly, we can show that the solution to the partial differential equation

$$\begin{cases} \frac{\partial p}{\partial t}(t, x) + Ap(t, x) - q(t, x)p(t, x) = 0, & 0 < t < T, x \in \mathbb{R} \\ p(t, x) = \phi(x), & x \in \mathbb{R} \end{cases}$$

(3.16) is given by

$$\begin{aligned} p(t, x) &= E^{t,x} \left[\phi(X(T)) \exp \left(- \int_t^T q(s, X(s))ds \right) \right] \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(T^{2H}-t^{2H})}} \exp \left(\frac{-(x-W_H(t))^2}{2(T^{2H}-t^{2H})} \right) p(x)dx. \end{aligned} \quad (3.17)$$

So that

$$\begin{aligned} P(t, x) &= E \left[x \exp \left(bW_H(t) + at - \frac{1}{2} b^2 t^{2H} k \right) - \right. \\ & \left. k^+ \exp(-a(T-t)) \right] \end{aligned} \quad (3.18)$$

Corollary 3.1: Let $H > \frac{1}{2}$ and $f: \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function with bounded derivatives. Define $P(x, t) = E_H[f(x + W_t)]$, we have

$$\frac{dP}{dt} + H\sigma^2 t^{2H-1} \bar{x}^2 \frac{\partial^2 P}{\partial x^2} = 0. \quad (3.19)$$

Proof:

Alternatively one can solve equation (3.2) for stock which is already priced in the market. To do this, we remove the effect of the discount rate r by letting $\bar{P} = e^{-rt}P \Rightarrow P = \bar{P}e^{rt}$, $\bar{x} = e^{-rt}x \Rightarrow x = \bar{x}e^{rt}$, (3.20) so that equation (3.2) becomes as required.

IV. THE MELLIN TRANSFORMATION

The Mellin transformation is a basic tool for analysing the behaviour of many important functions in mathematics and mathematical physics, such as the zeta functions occurring in number theory and in connection with various spectral problems. We describe it first in its simplest form and then explain how this basic definition can be extended to a much wider class of functions, important for many applications.

Let $\varphi(t)$ be a function on the positive real axis $t > 0$ which is reasonably smooth (actually, continuous or even piecewise continuous would be enough) and decays rapidly at both 0 and ∞ , i.e., the function $\varphi(t)$ is bounded on \mathbb{R}_+ for any $A \in \mathbb{R}$. Then the integral

$$\bar{\varphi}(s) = \int_0^\infty \varphi(t) t^{s-1} dt \quad (4.1)$$

converges for any complex value of s and defines a holomorphic function of s called the Mellin transform of $\varphi(s)$. The following small table, in which α denotes a complex number and λ a positive real number, shows how $\bar{\varphi}(s)$ changes when $\varphi(t)$ is modified in various simple ways:

$$\begin{array}{cccc} \varphi(\lambda t) t^\alpha \varphi(t) & \varphi(t^\lambda) & \varphi(t^{-1}) & \varphi'(t) \\ \lambda^{-s} \bar{\varphi}(s) \bar{\varphi}(s + \alpha) & \lambda^{-s} \bar{\varphi}(\lambda^{-1}s) & \bar{\varphi}(-s) & (1-s) \bar{\varphi}(s-1) \end{array} \quad (4.2)$$

We also mention, although we will not use it in the sequel, that the function $\varphi(t)$ can be recovered from its Mellin transform by the inverse Mellin transformation formula

$$\varphi(t) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \bar{\varphi}(s) t^{-s} ds, \quad (4.3)$$

where C is any real number. (That this is independent of C follows from Cauchy's formula).

However, most functions which we encounter in practice are not very small at both zero and infinity. If we assume that $\varphi(t)$ is of rapid decay at infinity but grows like t^{-A} for some real number A as $t \rightarrow 0$, then the integral (1) converges and defines a holomorphic function only in the right half-plane $\Re(s) > A$. Similarly, if $\varphi(t)$ is of rapid decay at zero but grows like t^{-B} at infinity for some real number B , while if $\varphi(t)$ has polynomial growth at both ends, say like t^{-A} at 0 and like t^{-B} at ∞ with $A < B$, then $\bar{\varphi}(s)$ is holomorphic only in the strip $A < \Re(s) < B$. But it turns out that in many cases the function $\bar{\varphi}(t)$ has a meromorphic extension to a larger half-plane or strip than the one in which the original integral (1) converges, or even to the whole complex plane. Moreover, this extended Mellin transform can sometimes be defined even in cases where $A > B$, in which case the integral (4.3) does not converge for any value of s at all.

Taking the Mellin transform of the fractional Black-Scholes partial differential equation for an option price given in (3.19), we have that

$$M\left(\frac{\partial P(x_t, t)}{\partial t}\right) = M\left(-H\sigma^2 x_t^2 t^{2H-1} \frac{\partial^2 P(x_t, t)}{\partial s_t^2}\right) \quad (4.4)$$

with the boundary condition

$$\left. \begin{aligned} p(S_t, T) &= \max[e^{-r(t-T)}(K - S_t), 0] \\ \lim_{S_t \rightarrow 0} p(S_t, T) &= Ke^{-r(t-T)} \\ \lim_{S_t \rightarrow \infty} p(S_t, T) &= 0 \end{aligned} \right\}. \quad (4.5)$$

Using the properties of the Mellin transforms, we have;

$$\left. \begin{aligned} M\left(\frac{\partial p}{\partial t}(x, t)\right) &= \frac{d}{dt} p(v, t) \\ M(H\sigma^2 t^{2H-1} x^2 P(s, t)) &= -H\sigma^2 t^{2H-1} (v^2 + v) P(v, t) \end{aligned} \right\}. \quad (4.6)$$

Substituting (4.6) into (4.4) and simplifying further yields

$$\frac{dp(v, t)}{dt} = -H\sigma^2 t^{2H-1} (v^2 + v) p(v, t), \quad t \in [0, T]. \quad (4.7)$$

Integrating (4.7) yields

$$p(v, t) = A(v) - \frac{\sigma^2 t^{2H}}{2} (v^2 + v). \quad (4.8)$$

$$\begin{aligned} & \int_{-\infty}^{\infty} \left| \left(\frac{K^{1+m+nj}}{(m+jn)(m+nj+1)} - \frac{\sigma^2}{2} (m+jn)^2 + (m+nj) \right) (t^{2H} - T^{2H}) \right| |x_t^{-(m+jn)}| |e^{-r(t-T)}| dn \\ & \leq M(m) x_T^{-m} \left(\frac{K^{1+m+nj}}{(m+jn)(m+nj+1)} - \frac{\sigma^2}{2} (m+jn)^2 + (m+nj) \right) (t^{2H} - T^{2H}) \int_{-\infty}^{\infty} \exp(-rn(t-T)) dn \end{aligned} \quad (4.20)$$

Using the differentiation theorem of parameter integrals and the fact that

$$\int_{-\infty}^{\infty} n^j \exp\left(\frac{-\sigma^2}{2}\right) (T-t) dn < \infty, \quad j = 0, 1, 2, \dots, \quad t \in [0, T]. \quad (4.21)$$

Then it follows that upon differentiation of (4.17), we have that

Setting

$$\psi(v) = \frac{\sigma^2 t^{2H}}{2} (v^2 + v), \quad (4.9)$$

then (4.8) becomes

$$p(v, t) = A(v) - \psi(v). \quad (4.10)$$

Where $A(v)$ is a constant of integration to be determined and it is defined as

$$A(v) = \psi(v, t) + \frac{\sigma^2}{2} T^{2H} (v^2 + v) \quad (4.11)$$

$\psi(v, t)$ can be obtained by taking the Mellin transform of the initial condition of the form

$$\psi(v, t) = \int_0^{\infty} \max(k - x_t)^+ x_t^{v-1} dx_t = \frac{K^{1+v}}{v(v+1)}. \quad (4.12)$$

Using equations (4.10), (4.11) and (4.12), we have that

$$p(v, t) = \frac{K^{1+v}}{v(v+1)} - \frac{\sigma^2}{2} (v^2 + v) (t^{2H} - T^{2H}). \quad (4.13)$$

Combining (3.20) and (4.13) we have;

$$\bar{P} = \left[\frac{K^{1+v}}{v} - \frac{\sigma^2}{2} \psi(r) (t^{2H} - T^{2H}) \right] e^{-r(t-T)}. \quad (4.14)$$

The Mellin inversion of (4.14) is obtained as

$$\begin{aligned} M^{-1}(p(v, t)) &= p(x_t, t) = \frac{1}{2\pi j} \int_{a-j\infty}^{a+j\infty} \left[\frac{K^{1+v}}{v(v+1)} - \frac{\sigma^2}{2} (v^2 + v) (t^{2H} - T^{2H}) \right] e^{-r(t-T)} x_t^{-v} dv. \end{aligned} \quad (4.15)$$

In what follows, we want to show that the expression (4.15) is a solution of the Black-Scholes partial differential equation for options price given by (3.2). Assume that

$$v = m + jn \Rightarrow dv = jdn \quad (4.16)$$

Substituting (4.16) into (4.15) yields

$$\begin{aligned} p(x_t, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{K^{1+m+nj}}{(m+jn)(m+nj+1)} - \frac{\sigma^2}{2} (m+jn)^2 + (m+nj) \right] (t^{2H} - T^{2H}) e^{-r(t-T)} x_t^{-(m+jn)} dn. \end{aligned} \quad (4.17)$$

But $P(x_t, t)$ is Mellin transformable and continuous, therefore setting $t = T$, we have (4.17) becoming

$$P(x_t, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{K^{1+m+nj}}{(m+jn)(m+jn+1)} x_t^{-(m+jn)} dn. \quad (4.18)$$

Equation (4.17) is well defined and satisfies (4.18).

Using the definition of the Mellin transforms, then

$$\left| \frac{K^{1+m+nj}}{(m+jn)(m+jn+1)} \right| \leq M(m) \int_0^{\infty} |f(s)| s^{m-1} ds \quad \forall n \in R \quad (4.19)$$

and for $t \in [0, T]$ we have that

$$\left. \begin{aligned} \frac{\partial P(x_t, t)}{\partial t} &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{K^{1+m+nj}}{(m+jn)(m+nj+1)} - \frac{\sigma^2}{2} (m+jn)^2 + (m+nj) \right] (t^{2H} - T^{2H}) e^{-r(t-T)} x_t^{-(m+jn)} dn \\ \frac{\partial^2 P(x_t, t)}{\partial x_t^2} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{K^{1+m+nj}}{(m+jn)(m+nj+1)} - \frac{\sigma^2}{2} (m+jn)^2 + (m+nj) \right] (t^{2H} - T^{2H}) e^{-r(t-T)} x_t^{-(m+jn+2)} dn \end{aligned} \right\} \quad (4.22)$$

Substituting (4.22) into the Black-Scholes partial differential equation for the option price given in (3.19)

$$\begin{aligned} \frac{d\bar{P}}{dt} + H\sigma^2 t^{2H-1} \bar{x}^2 \frac{\partial^2 \bar{P}}{\partial \bar{x}^2} \\ = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{K^{1+m+nj}}{(m+jn)(m+nj+1)} - \frac{\sigma^2}{2} (m+jn)^2 \right. \\ \left. + (m+nj)(t^{2H} - T^{2H}) \right) e^{-r(t-T)} x_t^{-(m+jn)} dn \\ + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{K^{1+m+nj}}{(m+jn)(m+nj+1)} - \frac{\sigma^2}{2} (m+jn)^2 \right. \\ \left. + (m+nj)(t^{2H} - T^{2H}) \right) e^{-r(t-T)} x_t^{-(m+jn+2)} dn \\ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \left(\frac{K^{1+m+nj}}{(m+jn)(m+nj+1)} - \frac{\sigma^2}{2} (m+jn)^2 \right. \right. \\ \left. \left. + (m+nj)(t^{2H} - T^{2H}) \right) e^{-r(t-T)} \right\} \\ \times (x_t^{-(m+jn+2)} - x_t^{-(m+jn)}) dn = 0. \end{aligned}$$

Hence $p(x_t, t)$ defined by (4.15) is a solution of (3.9).

V. CONCLUSION

We have suggested an alternative approach in the derivation of the fractional Black-Scholes PDE for the pricing of options (order than those found in literature, see Osu and Chukwunezu [27] and reference therein) and solution proffered using Mellin transform. It is worthy to note that to use the Mellin transform and condition that guarantee its existence, we assume $p(x_t, t)$ is bounded of polynomial degree when $x_t \rightarrow 0$ and $x_t \rightarrow \infty$.

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