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Pseudo-metric Space and Its Properties

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***Abstract* – In this paper, we design pseudo-metric space based on the concept of metric space, and give the concepts of pseudo spherical neighbourhood and pseudo open set and pseudo neighbourhood in the pseudo-metric Space. Based on the concept of pseudo spherical neighbourhood, the conclusions of open set and neighbourhood can be extended to pseudo metric space. Then we give proofs of the relationships between pseudo-metric space and the property of countable spaces in terms of these definitions.**

***Keywords* – Pseudo-metric Space; Pseudo Spherical Neighbourhood; Pseudo Neighbourhood; Separable Space.**

I. Introduction

Sphere neighborhood plays an important role in the study of the properties of metric spaces. In general, the function of spherical neighborhood in metric space is equivalent to open sets in Topological Spaces. In order to consider the properties of pseudo-metric space, we define a pseudo spherical neighborhood. At present, experts and scholars of home and abroad have researched pseudo-metric space, and have given their own conclusions. For example, Li Bijing ([1] [2]refs) studied on logic pseudo-metric space and came into a conclusion--logical formulas are measurable, in the logic system MTL induced by left-continuous triangle module; Wang Xiaojuan ([3] refs) discussed the problem about every cover of X exist *Lebesgue* number on a compact pseudo-metric space (X,d);He Gang ([4] refs) set up the Baire Theorem of Category in pseudo-metric space; Hui Xiaojing ([5] refs) gave the definition of probability logic pseudo-metric spaces and it's properties; Liu Mingxue ([6] refs) studied the relations between the random pseudo-metric collection space and the probabilistic pseudo-metric collection space. There are many practical features on a compact pseudo-metric spaces. In fact, spherical neighbourhood plays a vital role in metric space. Thanks to that foundation, we introduce pseudo spherical neighbourhood on a compact pseudo-metric spaces, besides, the properties of metric spaces can be popularized to pseudo-metric spaces by pseudo spherical neighbourhood, and then we can discuss the relationship between pseudo-metric spaces and the property of countable spaces.

We can discuss many useful properties of pseudo-metric space by different ways. Based on the concept of pseudo spherical neighbourhood, the conclusions of open set and neighbourhood can be extended to pseudo metric space. Next we can also discuss the relationships between pseudo-metric space and the property of countable spaces in terms of these definitions.

II. Pseudo-metric Space

Metric space is widely used in mathematics, and pseudo-metric space is a generalization of metric spaces. That is to say, metric space is a special form of pseudo metric space. Here is the accepted mathematical definition of an abstract distance.

Definition 1.1[7] A metric space is a set X, together with a metric (also called distance)

$$d: X\times X \to [0, \infty)$$

satisfying the following properties:

$(a)\ d(x,y) = 0$ *if and only if* $x = y$;

$(b)\ d(x,y) = d(y,x)$;

$(c)\ d(x,y) \le d(x,z) + d(z,y)$.

Example 1.1[8] On any set X, the function

$$d(x,y)=\begin{cases}0, & if\ x=y,\\ 1, & if\ x\neq y.\end{cases}$$

is a metric, called the *discrete metric*.

Definition 1.2[9] A pseudo-metric space is a pair (X, d) where X is a set and d is a function

$$d: X\times X \to [0, \infty),$$

called a pseudo-metric, that satisfies the following properties for all x, y, z in X:

$(a)\ d(x,y)=0$ *if and only if* $x=y$;

$(b)\ d(x,y)=d(y,x)$;

(c) *pseudo triangle inequality*: $\exists \alpha \ge 1$,

$s.t.\ d(x,y) \le \alpha d(x,z)+d(z,y)$.

According to the definitions of the metric space and pseudo-metric space, we found that reflexivity and symmetry are needed to be satisfied whether it is in a metric space or pseudo-metric space. The different place between the two space is that pseudo triangle inequality is the generalization of the triangle inequality, thus metric space is a special form of pseudo-metric space.

Proposition 1 Pseudo-metric space is a generalization of metric space, metric space is pseudo-metric space, but pseudo-metric space is not necessarily a metric space.

Proof. Let (X,d) be a metric space, and $\alpha \ge 1$, thus $d(x,y)\le d(x,z)+d(z,y)\le \alpha d(x,z)+d(z,y)$ That is, there is a $\alpha \ge 1$, and $x,y,z\in X$, so we get that $d(x,y)\le \alpha d(x,z)+d(z,y)$.Thus d is the pseudo-metric in X ,and (X,d) is a pseudo-metric space.

Pseudo-metric space is not necessarily a metric space, we only give a counterexample.

Let $X=\{1,2,3\}$, define d:

$$\begin{cases} d(1,1)=d(2,2)=d(3,3)=0 \\ d(1,2)=d(2,1)=d(2,3)=d(3,2)=1 \\ d(1,3)=d(3,1)=4 \end{cases}$$

Obviously, d meets the reflexivity and symmetry. Therefore, it is necessary to verify the pseudo triangle inequality.

Let $\alpha=3$, we can find that for arbitrary $x,y,z\in X$,we get that $d(x,y)\le \alpha d(x,z)+d(z,y)$ .Thus (X,d) is a pseudo-metric space, but $d(1,3)=4>d(1,2)+d(2,3)=2$,so (X,d) is not a metric space.

III. Math Pseudo Spherical Neighborhood

Spherical neighborhood plays an important role in the study of metric space's properties. Generally speaking, the spherical neighborhood in metric space is equivalent to the open sets in a topological space. In order to further study the properties of pseudo-metric spaces, we define pseudo-metric space "sets" - a pseudo spherical neighborhood.

Definition 2.1 Let (X,d) be a pseudo-metric space, $x\in X$, the (open) ball of radius $\varepsilon>0$ centered at $x\in X$ is

$$B^*(x,\varepsilon)=\{y\in X|d(x,y)<\varepsilon\}$$

called the pseudo spherical neighborhood.

Theorem 2.1 Pseudo spherical neighborhood of pseudo-metric space has the following basic properties:

(1) Each point $x\in X$ has at least one pseudo sphere neighborhood, and the point x belongs to each of its pseudo spherical neighborhood;

(2) In the presence of an arbitrary two pseudo spherical neighborhood of the point $x\in X$, there exists a pseudo spherical neighborhood is included the two pseudo spherical neighborhood;

(3) If $y\in X$ belongs to a pseudo spherical neighborhood of $x\in X$, then there exists a pseudo spherical neighborhood of $y\in X$ is contained in the pseudo spherical neighborhood of x.

Proof. (1) Let $x\in X$, for any $\varepsilon>0$, $\mathrm{B}^*(x,\varepsilon)$ is a pseudo spherical neighborhood of x ,then x has at least one pseudo spherical neighborhood; because of $d(x,x)=0$,so x belongs to its each pseudo spherical neighborhood.

(2) For any $x\in X$, if there are two pseudo spherical neighborhoods of x, record as $\mathrm{B}^*(x,\varepsilon_1)$ and $\mathrm{B}^*(x,\varepsilon_2)$, for every $\varepsilon>0$,then $\varepsilon<\min\{\varepsilon_1,\varepsilon_2\}$,we can get $\mathrm{B}^*(x,\varepsilon)\subset \mathrm{B}^*(x,\varepsilon_1)\cap \mathrm{B}^*(x,\varepsilon_2)$,so the $\mathrm{B}^*(x,\varepsilon)$ is included the two pseudo spherical neighborhood.

(3) Let $y\in \mathrm{B}^*(x,\varepsilon)$ and $\varepsilon_1=\dfrac{\varepsilon-d(x,y)}{\alpha}>0$. If $z\in \mathrm{B}^*(y,\varepsilon_1)$, then $d(z,x)\le \alpha d(z,y)+d(y,x)<\alpha\varepsilon_1+d(y,x)=\varepsilon$, so $z\in \mathrm{B}^*(x,\varepsilon)$. That is to say, $\mathrm{B}^*(y,\varepsilon_1)\subset \mathrm{B}^*(x,\varepsilon)$.

Definition 2.2 Let (X,d) be a pseudo-metric space, and let A be a subset of X .For each point of A ,there exists a pseudo spherical neighborhood that is included by A ,we call A as a pseudo open set of pseudo metric space.

According to theorem 1.1 (3), we find that every pseudo spherical neighborhood is pseudo - open set. In order to facilitate the discussion, we will promote the concept of pseudo spherical neighborhood.

Definition 2.3 Let (X,d) be a pseudo-metric space, $x\in X$,and let U be a subset of X .If there is a pseudo open set V satisfying the conditions $x\in V\subset U$, U is called a pseudo neighborhood of x.

The following theorem gives an equivalent statement of the pseudo neighborhood, and show that it is natural to generalize from the pseudo spherical neighborhood to the pseudo neighborhood.

Theorem 2.2 Let (X,d) be a pseudo-metric space, $x\in X$,and let U be a subset of X. U is a pseudo neighborhood of x if and only if there is a pseudo spherical neighborhood contained in U.

Proof. (Necessity) U is a pseudo neighborhood of x ,according to the definition of pseudo neighborhood, there is a pseudo open set V satisfying $x\in V\subset U$. According to the definition of pseudo open set, there is a pseudo spherical neighborhood of x contained in *V*, then the pseudo spherical neighborhood also contained in *U*. That is, there is a pseudo spherical neighborhood of x contained in U.

(Adequacy) If there is a pseudo spherical neighborhood of x contained in U, because a pseudo spherical neighborhood is pseudo open set, that is, U is a pseudo neighborhood of x.

IV. The Relations Between Pseudo Metric Spaces and Countability Axioms

With the definition of the pseudo spherical neighborhood, we explore the relationship between the pseudo metric space and the countability axioms.

Theorem 3.1[10] Any pseudo metric space is the space that satisfies the first countability axiom.

Proof. Let (X,d) be a pseudo-metric space, $x\in X$ .All of the pseudo spherical neighborhood formed in the center of x and the radius with a rational number constitute a countable neighborhood basis of x.

Theorem 3.2 If (X,d) is a separable pseudo-metric space, then (X,d) is the space that satisfies the second countability axiom.

Proof. Let (X,d) be a separable pseudo-metric space, and let D be a countable dense subset in X. Let $\beta=\left\{B^{*}\left(x,\frac{1}{n}\right) \middle| x\in D, n\in Z_{+}\right\}$.

It is clear that β is a countable family formed by the open set in X.

For any $y\in X$, let U be a pseudo neighborhood of y, then there is a $k\in Z_{+}$ satisfying $B^{*}\left(y,\frac{1}{k}\right)\subset U$.

Because D is a countable dense subset in X, then $B^{*}\left(y,\frac{1}{2k}\right)\cap D\neq\phi$.

For arbitrary $\tilde{y}\in B^{*}\left(y,\frac{1}{2k}\right)\cap D$, if $x\in B^{*}\left(\tilde{y},\frac{1}{2k}\right)\cap D$,

then $d(x,\tilde{y})<\frac{1}{2kM}$, so

$d(x,y)\le \alpha d(x,\tilde{y})+d(\tilde{y},y)<\frac{1}{k}$, that

is, $x\in B^{*}\left(y,\frac{1}{k}\right)$. Therefore, we get that

$B^{*}\left(\tilde{y},\frac{1}{2k}\right)\subset B^{*}\left(y,\frac{1}{k}\right)\subset U$, for $\tilde{y}\in D$, so

$B^{*}\left(\tilde{y},\frac{1}{2k}\right)\in\beta$.

Above all, we have proofed that for any $y\in X$ and every pseudo neighborhood U of y, there is a $B^{*}\left(\tilde{y},\frac{1}{2k}\right)\in\beta$.

Thus β is a basis of X, and (X,d) is the space that satisfies the second countability axiom.

V. Conclusion

This paper mainly defined pseudo spherical neighborhood in the pseudo-metric space, and gave based on the concept of pseudo - open sets and pseudo neighborhood and metric spaces and discussing the relationship between countability axioms and the pseudo-metric space. The relationship between separation axioms and the pseudo-metric space remains to be further studied.

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