International Journal of Applied Science and Mathematics
Volume 3, Issue 6, ISSN (Online): 2394-2894

A Class of Irreducible Modules over Witt Algebra $W_n (n > 1)$

Haijun Tan*

Abstract – **In this paper, by taking tensor products of modules over W_n we obtain a class of new irreducible modules over W_n.**

Keywords – **Witt Algebra, Non-weight Module, Irreducible Module.**

I. Introduction

Throughout this paper, we denote by $\mathbb{Z}, \mathbb{Z}_+, \mathbb{N}, \mathbb{C}$ and $\mathbb{C}^*$ the sets of all integers, non-negative integers,positive integers, complex numbers and non-zero complex numbers, respectively. All vectorspaces are over $\mathbb{C}$.

For $n \in \mathbb{N}$, let $A_n = \mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]$ bethe Laurent polynomial algebra over $\mathbb{C}$. The*Witt algebra of rank* n is theLie algebra of all derivations of A_n, and is denoted by W_n, that is,$W_n = Der(A_n)$. Denote by$\partial_i = t_i \frac{\partial}{\partial_i}$.For$r = (r_1, \cdots, r_n) \in \mathbb{Z}^n$ and $u = (u_1, \cdots, u_n) \in \mathbb{C}^n$, let $t^r = t^{r_1} \cdots t^{r_n}, D(u, r) = t^r \sum_{i=1}^n u_i \partial_i$.Then $W_n$is thelinear span of the set $\{D(u, r) | u \in \mathbb{C}^n, r \in \mathbb{Z}^n\}$.The Liebracket in $W_n$is defined by $[D(u, r), D(v, s)] = D(w, r + s)$ where $u, v \in \mathbb{C}^n, r, s \in \mathbb{Z}^n, w = (u|s)v - (v|r)u$ and $(\cdot|\cdot)$ is the standard symmetric bilinear form on$\mathbb{C}^n$.It is known that $\mathfrak{h} = \oplus_{i=1}^n \mathbb{C}\partial_i$ is the *Cartansubalgebra* of W_n. A $W_n$module V is called a *weight module* provided that the action of $\mathfrak{h}$ on V is diagonalizable. If the action of $\mathfrak{h}$ on $\mathfrak{h}$ is not diagonalizable, $\mathfrak{h}$ is a *non-weight module*.

The weight representation theory over W_n is well studied,especially, Billig and Futorny classified the Harish-Chandra modules over W_n in [1]. There are also examples of weight modules with weight spaces to be infinite-dimensional (see [2]). In [4], a lot of non-weight modules over W_n are constructed and studied, and one class of them are defined as follows.

Example 1.*Let $\mathbb{C}[x_1, \cdots, x_n]$ be the polynomial associative algebra over $\mathbb{C}$ in n commuting indeterminates $x_1, \cdots, x_n$. For any $a \in \mathbb{C}$ and $\Lambda = (\lambda_1, \cdots, \lambda_n) \in (\mathbb{C}^*)^n$ we define the action of$W_n$on$\mathbb{C}[x_1, \cdots, x_n]$by*

$$t^r \partial_i \cdot f(x_1, \cdots, x_n) = \Lambda^r (x_i - r_i a_i) f(x_1 - r_1, \cdots, x_n - r_n)$$

where $r = (r_1, \cdots, r_n) \in \mathbb{Z}^n, f(x_1, \cdots, x_n) \in \mathbb{C}[x_1, \cdots, x_n]$ and$\Lambda^r = \lambda_1^{r_1} \cdots \lambda_n^{r_n}$.The corresponding W_n-module is denoted by$\Omega(\Lambda, a)$. From [4] we know that$\Omega(\Lambda, a)$is irreducible if and only if$\neq 0$.

The main purpose of the present paper is to construct new irreducible W_n-modules by taking tensor products of finite W_n-modules$\Omega(\Lambda, a)$. More precisely, we will prove the following

Theorem 2. *Let$m \in \mathbb{N}$. For$\Lambda_j = (\lambda_{1j}, \cdots, \lambda_{nj}) \in (\mathbb{C}^*)^n, a_j \in \mathbb{C}^*, 1 \le j \le m$ with $\lambda_{ji} \neq \lambda_{j'i}, j \neq j', 1 \le i \le n$.let$\Omega(\Lambda_j, a_j)$ be the irreducible W_n-modules as in Example 1. Then the tensor product*

$$\Omega(\Lambda_1, a_1) \otimes \cdots \otimes \Omega(\Lambda_m, a_m)$$

is an irreducible W_n-module.

This theorem will be proved in Section II. This theorem tells us that from the known irreducible W_n-modules $\Omega(\Lambda_j, a_j)$ we can construct irreducible W_n-modules as long as the $\Lambda_j's$ satisfy the conditions mentioned in the theorem. Clearly, these modules are not isomorphic to the modules obtained in [4]. So **Theorem2** provides a method of finding irreducible modules over Witt algebra $W_n (n > 1)$.

II. Proving Theorem2

In this section, we will prove *Theorem 2*. To prove the theorem, we need the following crucial lemma which is Lemma 2 in [3].

Lemma 3. *Let$\lambda_1, \cdots, \lambda_m \in \mathbb{C}, s_1, \cdots, s_n, s \in \mathbb{N}$ with $s_1 + \cdots + s_n = s$.Define a sequence of functions on$\mathbb{Z}$as follows :*

$$f_1(n) = \lambda_1^n, f_2(n) = n\lambda_1^n, \cdots, f_{s_1}(n) = n^{s_1}\lambda_1^n,$$
$$f_{s_1+1}(n) = \lambda_2^n, \cdots, f_{s_1+s_2}(n) = n^{s_2-1}\lambda_2^n,$$
$$\cdots, f_s(n) = n^{s_m-1}\lambda_m^n.$$

Let $Y = (y_{pq})$ be the $s \times s$ matrix with $y_{pq} = f_q(p-1), q = 1, \cdots, s, p = r+1, \cdots, r+s$ where $r \in \mathbb{Z}_+$. Then

$$det(Y) = \prod_{j=1}^m (s_j - 1)!! \lambda_j^{s_j(s_j+2r-1)/2} \prod_{1 \le i < j \le m} (\lambda_i - \lambda_j)^{s_i s_j}$$

where $s_j!! = s_j! \times (s_{j-1}!) \times \cdots \times 1!$ with $0!! = 1$, for convenience.

Now we can prove our main theorem.

Proof of Theorem 2. For convenience, we denote by $M = \Omega(\Lambda_1, a_1) \otimes \cdots \otimes \Omega(\Lambda_m, a_m)$.Let 1_jbe the identity element of $\Omega(\Lambda_j, a_j)$and denote by

$$\bar{1} = 1_1 \otimes \cdots \otimes 1_m \in \mathrm{M}.$$

For $K_j = (k_{j1}, \cdots, k_{jn}) \in \mathbb{Z}_+^n$we denote by $X_j^{K_j} = \prod_{i=1}^n x_{ji}^{k_{ji}}$and denote by$X_1^{K_1} \cdots X_m^{K_m} = X_1^{K_1} \otimes \cdots \otimes X_m^{K_m}$. We define the *degree* of $X_1^{K_1} \cdots X_m^{K_m}$to be$\sum_{j=1}^m |K_j|$ where$|K_j| = \sum_{i=1}^n k_{ji}$. For $0 \neq f \in M$, the *degree* of f is defined to be the degree of its terms with maximal degree and is denoted by $\deg(f)$. Note that $\deg(\bar{1}) = 0$.

Firstly, we have the following

Claim 1. $\bar{1}$generates the whole module M.

Let W be the submodule of M generated by $\bar{1}$. We need only to show$X_1^{K_1} \cdots X_m^{K_m} \in W$for all $K_1, \cdots, K_m \in \mathbb{Z}_+^n$.For fixed $K_j \in \mathbb{Z}_+^n, 1 \le j \le m$,noting that

* H. Tan is partially supported by NSF of Jilin Province (Grant 20160520111JH)

$$t_i^{r_i}\partial_i \cdot X_1^{K_1}\cdots X_m^{K_m} = \sum_{j=1}^{m} X_1^{K_1}\cdots\left(t_i^{r_i}\partial_i\cdot X_j^{K_j}\right)\cdots X_m^{K_m}$$
$$= \sum_{j=1}^{m} X_1^{K_1}\cdots\left(\lambda_{ji}^{r_i}x_{j1}^{k_{j1}}\cdots(x_{ji} - r_i a_i)(x_{ji}-r_i)^{k_{ji}}\cdots x_{jn}^{k_{jn}}\right)\cdots X_m^{K_m}$$
$$= (\sum_{j=1}^{m}\lambda_{ji}^{r_i}x_{ji})X_1^{K_1}\cdots X_m^{K_m} + A$$

where A has degree $\sum_{j=1}^{m}|K_j|$. Taking $r_i = 0,1,\cdots,m-1$ and $i = 1,2,\cdots,n$,respectively, and by induction on $\sum_{j=1}^{m}|K_j|$ we see that $X_1^{K_1}\cdots X_m^{K_m} \in W$ for all $K_1,\cdots,K_m \in \mathbb{Z}_+^n$.Thus the claim holds.

The claim means that to prove the theorem is to prove that $\bar{1} \in W$ for any nonzero submodule W. Let W be a nonzero submodule of M and let $0 \neq f \in M$ with minimal degree. We have

Claim 2. $\deg(f) = 0$,thatis, $f = c\bar{1}$for some $c \in \mathbb{C}^*$.

To the contrary, assume $\deg(f) > 0$ For finding a contradiction we define a *partial order* on $\mathbb{Z}_+^n$ by

$$(k_1,\cdots,k_n) < (r_1,\cdots,r_n) \Leftrightarrow (\exists i \geq 0)(k_j = r_j,\ \forall j < i)(k_i < r_i)$$

where $(k_1,\cdots,k_n),(r_1,\cdots,r_n) \in \mathbb{Z}_+^n$, and define a *partial order* on the set $\{X_1^{K_1}\cdots X_m^{K_m} | K_1,\cdots,K_m \in \mathbb{Z}_+^n\}$by

$$X_1^{K_1}\cdots X_m^{K_m} < X_1^{R_1}\cdots X_m^{R_m} \Leftrightarrow (\exists i \geq 0)(K_j = R_j,\ \forall j < i)\ (K_i < R_i)$$

where $K_j, R_j \in \mathbb{Z}_+^n, 1 \leq j \leq m$.

Denote by $f = \sum_{\bar{K}\in S} c_{\bar{K}} X^{\bar{K}}$ where S is a finite subset of $\mathbb{Z}_+^{n\times m}$ and $\bar{K} = (K_1,\cdots,K_m) \in S$ with $K_j \in \mathbb{Z}_+^n, 1 \leq j \leq m$, $c_{\bar{K}} \in \mathbb{C}^*$ and $X^{\bar{K}} = X_1^{K_1}\cdots X_m^{K_m}$. Let $c_{\bar{K}_0}X_1^{K_{01}}\cdots X_m^{K_{0m}}$be the maximal nonzero term of $f$with respect to the partial order defined above. Clearly, $K_{0j} = (k_{0j1},\cdots,k_{0jn}) > 0$ for some j.So some $k_{0ji} > 0$. Letting $t_i^{r_i}\partial_i$act on f, we have

$$t_i^{r_i}\partial_i \cdot f = \sum_{\bar{K}\in S}\sum_{j=1}^{m} c_{\bar{K}} X_1^{K_1}\cdots\left(t_i^{r_i}\partial_i\cdot X_j^{K_j}\right)\cdots X_n^{K_n}$$
$$= \sum_{\bar{K}\in S}\sum_{j=1}^{m}\sum_{l=0}^{k_{ji}+1} r_i^l\lambda_{ji}^{r_i} b_{\bar{K},j,l} X_1^{K_1}\cdots(x_{j1}^{k_{j1}}\cdots x_{ji}^{k_{ji}+1-l}\cdots x_{jn}^{k_{jn}})\cdots X_m^{K_m}$$

where $b_{\bar{K},j,l} = c_{\bar{K}}(-1)^l(\binom{k_{ji}}{l} + \binom{k_{ji}}{l-1}a_i)$ and $r_i \in \mathbb{Z}_+$. From Lemma 3 we see that the coefficient of $r_i^l\lambda_{ji}^{r_i}$belongs to W for all j, l. In particular, the coefficient of $r_i^{k_{0ji}+1}\lambda_{ji}^{r_i}$ is

$$g = (-1)^{k_{0ji}+1}c_{\bar{K}_0}X_1^{K_{01}}\cdots(x_{j1}^{k_{0j1}}\cdots x_{ji}^{0}\cdots x_{jn}^{k_{0jn}})\cdots X_m^{K_{0m}} + \text{lower terms}$$

which has lower degree than $c_{\bar{K}_0}X_1^{K_{01}}\cdots X_m^{K_{0m}}$. This is contrary to the choice of f. Thus $deg(f) = 0$, as desired.

Claim 2 means that $\bar{1} \in W$ and the theorem follows from **Claim 1**.■

REFERENCES

[1] Y.Billig, and V. Futorny, Classification of irreducible representations of Lie algebra of vector fields on atorus, J. reine angel. Math., doi: 10.1515/crelle-2014-0059.

[2] G. Liu, and K.Zhao, New irreducible weight modules over Witt algebras with infinite-dimensional weight spaces, Bull. London Math. Soc. 47(2015), pp. 789-795.

[3] H. Tan, and K.Zhao, Irreducible modules from tensor products (II), J. Algebra, 394(2013), pp.357-373.

[4] H. Tan, and K. Zhao, Irreducible modules over Witt algebras W_n and over $sl_{n+1}(\mathbb{C})$, arXiv: 1312.5539.

AUTHOR'S PROFILE

Haijun Tan (1980-), Ph.D., graduated from Hebei Normal University (Shijiazhuang, Hebei, P. R. China) in 2014. The author's research area is the representation theory of Lie algebras. He works at Changchun University of Science and Technology. The author's published articles are as follows:

[1] H. Tan, and K. Zhao, Irreducible modules from tensor products, Ark. Mat., 54 (2016), pp. 181–200

[2] H. Tan, and K. Zhao, Irreducible modules from tensor products (II), J. Algebra, 394 (2013), pp. 357-373.

[3] H. Tan, and K. Zhao, W_n^+-and W_n-module structures on $U(\mathfrak{h}_n)$,J. Algebra, 424 (2015), pp. 357-375.